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THE ASYMPTOTIC FORM OF THE STRESSED STATE NEAR A THREE-DIMENSIONAL BOUNDARY SINGULARITY OF THE "CLAW" TYPE†

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Asymptotic formulae are derived for the fields of displacements, strains and stresses near a peak-shaped protrusion in the surface of an anisotropic elastic body (a "claw"-type singularity). The singular solutions constructed are interpreted as forces and torques concentrated at the tip of the peak, while the orders of growth of the displacement depend on the direction of the action of the force (longitudinal or transverse) and of the axis of the torque (twisting or bending) but not on the elastic properties of the material. The asymptotic analysis makes essential use of the observed analogy with one-dimensional models of thin rods of variable cross-section. © 2000 Elsevier Science Ltd. All rights reserved.

Determination of the singularities of the stressed state near irregularities of the boundary turns out to be a key factor in many branches of the mechanics of deformed bodies, such as the theory of fracture, computing methods and so on. Detailed studies have been devoted to corner and (to a lesser degree) conical spikes or inclusions for which the variables can be separated and the number of dimensions reduced. The geometrically most complicated singularities of the boundary require the development of new asymptotic methods for specific shapes (such as peak-shaped inclusions and cavities [5–7], as well as "beak-shaped" protrusions [8–9]). We stress that the specific properties of the system of equations of elasticity theory make it difficult to apply the mathematical tools suitable for scalar equations or systems of first-order equations—the necessary derivations and transformations become unmanageable. To reduce the number of dimensions in investigating "claw"-type singularities a decisive role has been played by the observed analogy with one-dimensional models of thin rods—to be precise, with the general asymptotic algorithms of [10, 11]. The formulae and the accompanying computations have been shortened by the use of matrix (rather than tensor) notation. In what follows a procedure for constructing formal asymptotic formulae will be presented; the legitimacy of the procedure follows from general results [12–15].

1. FORMULATION OF THE PROBLEM

Let Ω be a homogeneous anisotropic body whose surface $\partial\Omega$ has a singularity of the "claw" type (see Fig. 1), that is, near the origin O it coincides with a peak-shaped set

$$\{x=(y,\ z)\in \mathbf{R}^3: 0<z<d;\ z^{-\gamma}(y-Y(z))\in\omega\} \tag{1.1}$$

where $y=(y_1, y_2)$, $\gamma>1$ is a parameter characterizing the sharpness of the claw, ω is a two-dimensional domain bounded by a simple smooth closed contour, and the relations $y=Y(z)$, $z>0$, define the curved axis of the peak, which is tangent to the z axis; Y is a smooth vector-valued function, $Y(0)=0$, $\partial_z Y(0)=0$. By scaling, we reduce the geometric parameter d to unity, that is, we make the coordinates dimensionless.

We will write the elasticity problem in matrix form. The six-dimensional column vector of strains is defined as

$$\varepsilon(u)=(\varepsilon_{11},\ \varepsilon_{22},\ \varepsilon_{33},\ \alpha^{-1}\varepsilon_{23},\ \alpha^{-1}\varepsilon_{31},\ \alpha^{-1}\varepsilon_{13})^t \tag{1.2}$$

where ε_{ij} are the Cartesian components of the strain tensor, t is the transposition symbol, and the factor $\alpha=2^{-1/2}$ is chosen to ensure that the natural norms of the strain tensor and strain vector coincide. If the vector of displacements u is interpreted as a column vector $(u_1, u_2, u_3)^t$, then

$$\varepsilon(u)=D(\nabla)^t u$$

$$D(\nabla)=\begin{Vmatrix}\partial_1 & 0 & 0 & 0 & \alpha\partial_3 & \alpha\partial_2\\ 0 & \partial_2 & 0 & \alpha\partial_3 & 0 & \alpha\partial_1\\ 0 & 0 & \partial_3 & \alpha\partial_2 & \alpha\partial_1 & 0\end{Vmatrix},\quad \nabla=\text{grad},\ \ \partial_j=\frac{\partial}{\partial x_j}$$

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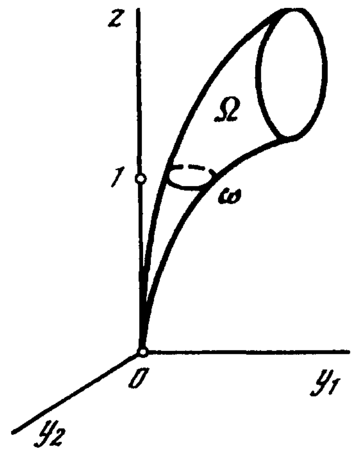

Fig. 1.

Let $\sigma(u)$ be the column vector of stresses analogous to (1.2) and let A be the symmetric positive-definite matrix of elasticity moduli. Hook's law in a matrix notation is

$$\sigma(u) = A\varepsilon(u) \tag{1.3}$$

The equations of equilibrium and the boundary conditions in stresses are

$$L(\nabla)u(x) \equiv D(-\nabla)AD(\nabla)^t u(x) = f(x), \quad x \in \Omega \tag{1.4}$$

$$B(x, \nabla)u(x) \equiv D(n(x))AD(\nabla)^t u(x) = g(x), \quad x \in \partial\Omega \setminus O \tag{1.5}$$

If $\nu = (\nu_1, \nu_2)^t$ is the normal to the contour $\partial\omega \subset \mathbf{R}^2$, then by (1.2) the following formulae hold near the point O for the normal n to $\partial\Omega$

$$n(x) = N(x)^{-1}(\nu_1(\eta(y, z)), \nu_2(\eta(y, z)), \nu_0(y, z))$$

$$N(x) = (1 + \nu_0^2(\eta(y, z)))^{1/2}, \quad \eta(y, z) = z^{-\gamma}(y - Y(z))$$

$$\nu_0(y, z) = -\nu_1(\eta(y, z))(\gamma z^{-1}(y_1 - Y_1(z)) + Y_1'(z)) -$$

$$-\nu_2(\eta(y, z))(\gamma z^{-1}(y_2 - Y_2(z)) + Y_2'(z))$$

The loads f and g are applied far from the point O. If they are in equilibrium, a solution u of problem (1.4), (1.5) exists in the energy class $\mathbf{W}^1_2(\Omega)^3$. Near the singularity this solution coincides apart from exponentially small terms with a rigid displacement (see below, Section 3). In the case of arbitrary f and g they may be balanced by forces applied at the point O (the claw "scratches" an absolutely rigid body). The aim of this paper is to construct the corresponding singular solutions and to determine their properties.

2. ASYMPTOTIC SOLUTION OF THE HOMOGENEOUS PROBLEM

Near the point O, we seek a formal asymptotic solution of problem (1.4), (1.5) as a power series:

$$u(x) = U^{-2}(z) + U^{-1}(y, z) + U^0(y, z) + U^1(y, z) + U^2(y, z) + \ldots \tag{2.1}$$

$$U^k(y, z) = z^{\varkappa_k} Q^k(\eta) \tag{2.2}$$

The unknowns are both the exponent $\varkappa_k$ and the factor Q^k, which is a function of the variables $\eta = z^{-\gamma}(y - Y(z))$.

Note that the matrices $D(\nabla)$ and $D(n(x))$ may be represented as follows:

$$D(\nabla) = D_y + D_z, \quad D(n) = D_\nu + \nu_0 D_1 \tag{2.3}$$

$$D_y = \begin{Vmatrix} \partial_1 & 0 & 0 & 0 & 0 & \alpha\partial_2 \\ 0 & \partial_2 & 0 & 0 & 0 & \alpha\partial_1 \\ 0 & 0 & 0 & \alpha\partial_2 & \alpha\partial_1 & 0 \end{Vmatrix}, \quad D_z = D_1\partial_z$$

$$D_v = \begin{Vmatrix} v_1 & 0 & 0 & 0 & 0 & \alpha v_2 \\ 0 & v_2 & 0 & 0 & 0 & \alpha v_1 \\ 0 & 0 & 0 & \alpha v_2 & \alpha v_1 & 0 \end{Vmatrix}, \quad D_1 = \begin{Vmatrix} 0 & 0 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{Vmatrix}$$

By (1.4), (1.5) and (2.3), we have

$$\begin{aligned} &L(\nabla) = L^0(\nabla_y) + L^1(\nabla_y, \partial_z) + L^2(\partial_z) \\ &N(x)B(x, \nabla) = B^0(y, z, \nabla_y) + B^1(y, z, \nabla_y, \partial_z) + B^2(y, z, \partial_z) \end{aligned} \tag{2.4}$$

$$\begin{aligned} &L^0 = -D_y A D_y^t, \; L^1 = -D_y A D_z^t - D_z A D_y^t, \; L^2 = -D_z A D_z^t \\ &B^0 = D_v A D_y^t, \; B^1 = D_v A D_z^t + v_0 D_1 A D_y^t, \; B^2 = v_0 D_1 A D_z^t \end{aligned} \tag{2.5}$$

The operators on the right of (2.4) possess the following generalized similarities:

$$\begin{aligned} &L^j(\nabla)(z^{\varkappa} U(\eta(y, z)) = z^{\varkappa - 2\gamma - j(1-\gamma)} F_{\varkappa}^j(\eta(y, z)) \\ &B^j(x, \nabla)(z^{\varkappa} U(\eta(y, z))) = z^{\varkappa - \gamma - j(1-\gamma)} G_{\varkappa}^j(\eta(y, z)) \end{aligned}$$

Formulae (2.4) are interpreted as expansions in powers of $z^{\gamma-1}$, and it is therefore logical to assume in (2.2) that $\varkappa_k = \varkappa_{k-1} + 1 - \gamma$.

Substituting (2.1) and (2.4) into (1.4), (1.5) and collecting coefficients of like powers of z, we obtain a recurrence sequence of problems for the "cross-section" $\omega(z)$

$$\begin{aligned} &L^0 U^j = -L^1 U^{j-1} - L^2 U^{j-2} \text{ in } \omega(z) \\ &B^0 U^j = -B^1 U^{j-1} - B^2 U^{j-2} \text{ on } \partial\omega(z) \end{aligned} \tag{2.6}$$

where $j = -2, -1, \ldots, U^{-4} = U^{-3} = 0$.

It is well known that when the right-hand sides are smooth the two-dimensional problem

$$L^0 U = F \text{ in } \omega(z), \; B^0 U = G \text{ on } \partial\omega(z) \tag{2.7}$$

has a solution U if and only if the following orthogonality conditions hold

$$(F, \Phi^q)_{\omega(z)} + (G, \Phi^q)_{\partial\omega(z)} = 0, \; q = 1, \ldots, 4 \tag{2.8}$$

$$\Phi^1 = e^1, \; \Phi^2 = e^2, \; \Phi^3 = e^3, \; \Phi^4(\eta) = \alpha(\eta_1 e^2 - \eta_2 e^1) \tag{2.9}$$

where $(\cdot,\cdot)$ is the scalar product in $L_2(\Xi)$ and e^j is the unit vector along the x^j axis. The smooth (bounded) solution U of problem (2.7) is defined apart from a linear combination of the vectors (2.9) and becomes unique if the condition $(U, \Phi^q) = 0$ $(q = 1, \ldots, 4)$ is satisfied

The structure of the initial terms of series (2.1) is the same as in the asymptotic expansion used in the theory of thin rods (see, e.g. [10])

$$\begin{aligned} &U^{-2}(z) = e^1 w_1(z) + e^2 w_2(z) \\ &U^{-1}(y, z) = w_3(z) e^3 + w_4(z) \Phi^4(\eta) - e^3(\eta_1 \partial_z w_1(z) + \eta_2 \partial_z w_2(z)) \end{aligned} \tag{2.10}$$

The functions w_p form a column vector $w = (w_1, \ldots, w_4)^t$ which is yet to be determined. Here w_1 and w_2 describe in the main the deflection of the claw, w_3 its displacement along the axis and w_4 its twisting.

According to (2.6), the problem for L^0 is

$$\begin{aligned} &L^0 U^0 = D_y A D_z^t U^{-1} + D_z A \{D_y^t U^{-1} + D_z^t U^{-2}\} \text{ in } \omega(z) \\ &B^0 U^0 = -D_v A D_z^t U^{-1} - v_0 D_1 A \{D_y^t U^{-1} + D_z^t U^{-2}\} \text{ on } \partial\omega(z) \end{aligned} \tag{2.11}$$

According to equalities (2.10), the sums in braces in (2.11) vanish. Thus, the right-hand sides of system (2.11) satisfy the conditions for (2.8) to be solvable.

Let us express the solution U^0 of problem (2.11) in a form convenient for further transformations. To that end, we note the equalities, which follow from (2.10) and (2.3)

$$D_z^t U^{-1}(y,\ z) = Y(y)\mathbf{D}(\partial_z)w(z) \tag{2.12}$$

$$\mathbf{D}(\partial_z) = \mathrm{diag}\{\partial_z^2,\ \partial_z^2,\ \partial_z,\ \partial_z\}$$

$$Y(y)^t = \begin{Vmatrix} 0 & 0 & -y_1 & 0 & 0 & 0 \\ 0 & 0 & -y_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha^2 y_1 & -\alpha^2 y_2 & 0 \end{Vmatrix}$$

By (2.11) and (2.12)

$$U^0(y,\ z) = X(y,\ z)\mathbf{D}(\partial_z)w(z) \tag{2.13}$$

The matrix-valued function X satisfies the conditions

$$L^0 X = D_y A Y \ \text{ in } \omega(z),\ B^0 X = -D_\nu A Y \text{ on } \partial\omega(z) \tag{2.14}$$

The vector U^1 is determined from the boundary-value problem

$$L^0 U^1 = D_y A D_z^t U^0 + D_z A\{D_y^t U^0 + D_z^t U^{-1}\} \text{ in } \omega(z)$$

$$B^0 U^1 = -D_\nu A D_z^t U^0 - \nu_0 D_1 A\{D_y^t U^0 + D_z^t U^{-1}\} \text{ on } \partial\omega(z) \tag{2.15}$$

The following formula holds for any function H smooth in $\overline{\Omega}$ (see, e.g. [16])

$$\frac{d}{dz}\int_{\omega(z)} H(y,\ z)dy = \int_{\omega(z)}\frac{\partial H}{\partial z}(y,\ z)dy - \int_{\partial\omega(z)} H(y,\ z)\nu_0(y,\ z)ds_y \tag{2.16}$$

Using this relation, one shows that the first two conditions (2.8) for problem (2.15) to be solvable are satisfied. We have

$$0 = (e^i,\ \partial_z D_1 A\{D_y^t U^0 + D_z^t U^{-1}\})_{\omega(z)} + (e^i,\ -\nu_0 D_1 A\{D_y^t U^0 + D_z^t U^{-1}\})_{\partial\omega(z)} =$$
$$= \partial_z(D_1^t e^i,\ A\{D_y^t U^0 + D_z^t U^{-1}\})_{\omega(z)} = \partial_z(-e^3 y_i,\ D_y A\{D_y^t U^0 + D_z^t U^{-1}\})_{\omega(z)} +$$
$$+\ \partial_z(e^3 y_i,\ D_\nu A\{D_y^t U^0 + D_z^t U^{-1}\})_{\partial\omega(z)} \tag{2.17}$$

where we have used the formula, which follows from (2.3)

$$D_1^t e^i = D_y^t e^3 y_i,\quad i = 1, 2 \tag{2.18}$$

The right-hand side of (2.18) vanishes by (2.11). By (2.13) and (2.16), the other two conditions for problem (2.15) to be solvable become

$$0 = (\Phi^i,\ D_z A\{D_y^t U^0 + D_z^t U^{-1}\})_{\omega(z)} - (\Phi^i,\ \nu_0 D_1 A\{D_y^t U^0 + D_z^t U^{-1}\})_{\partial\omega(z)} = \tag{2.19}$$
$$= -\partial_z(D_1^t \Phi^i,\ A\{D_y^t X + Y\})_{\omega(z)}\mathbf{D}(\partial_z)w(z),\ \ i = 3, 4$$

As a result, we obtain two equations for the vector w. One more pair of equations appears as a condition for the next problem to be solvable

$$L^0 U^2 = D_y A D_z^t U^1 + D_z A\{D_y^t U^1 + D_z^t U^0\} \text{ in } \omega(z)$$

$$B^0 U^2 = -D_\nu A D_z^t U^1 - \nu_0 D_1 A\{D_y^t U^1 + D_z^t U^0\} \ \text{ on } \partial\omega(z) \tag{2.20}$$

Taking the relationships (2.16), (2.18) and (2.15) into account, as well as (2.12) and (2.13), we have

$$0 = (e^i,\ D_z A\{D_y^t U^1 + D_z^t U^0\})_{\omega(z)} - (e^i,\ \nu_0 D_1 A\{D_y^t U^1 + D_z^t U^0\})_{\partial\omega(z)} =$$
$$= \partial_z(D_y y_i e^3,\ D_1 A\{D_y^t U^1 + D_z^t U^0\})_{\omega(z)} = \partial_z(e^i,\ D_1 A\{D_y^t U^1 + D_z^t U^0\})_{\omega(z)} =$$
$$= \partial_z[(y_i e^3,\ D_z A\{D_y^t U^0 + D_z^t U^{-1}\})_{\omega(z)} - (y_i e^3,\ \nu_0 D_1 A\{D_y^t U^0 + D_z^t U^{-1}\})_{\partial\omega(z)}] =$$
$$= -\partial_z^2(y_i D_1^t e^3,\ A\{D_y^t X + Y\})_{\omega(z)}\mathbf{D}(\partial_z)w(z) = 0,\ \ i = 1, 2 \tag{2.21}$$

The expressions $y_i D_1^i e^3$ and $D_1^i \Phi^i$ occurring in (2.21) and (2.19) are identical with the columns of the matrix Y. Hence the system of differential equations for determining w becomes

$$\mathbf{L}(z, \partial_z)w(z) \equiv \mathbf{D}(-\partial_z)\mathbf{M}(z)\mathbf{D}(\partial_z)w(z) = 0 \tag{2.22}$$

By (2.14), the 4×4 matrix $\mathbf{M}$ is a positive-definite and symmetric Gram matrix

$$\mathbf{M}(z) = \int_{\omega(z)} Y' A(D_y^i X + Y)dy = \int_{\omega(z)} (D_y^i X + Y)' A(D_y^i X + Y)dy \tag{2.23}$$

3. SPECIAL SOLUTIONS

It can be shown that system (2.22) has the following solutions

$$\mathbf{W}^q = \mathbf{e}^q, \quad q = 1, \ldots, 4; \quad \mathbf{W}^{4+i}(z) = z\mathbf{e}^i, \quad i = 1, 2 \tag{3.1}$$

where $\mathbf{e}^1, \ldots, \mathbf{e}^4$ are unit vectors along the axes in R^4. According to (2.10) and (2.12), if the column-vectors (3.1) are substituted into formula (2.1), the terms $U^2, U^3, \ldots$ vanish (for example, the operator $\mathbf{D}(\partial_z)$ nullifies $\mathbf{W}^1, \ldots, \mathbf{W}^6$ in (2.13)), while the terms of (2.10) with $\mathbf{W}^q$ instead of w form rigid displacements $\Phi^q(y, z)$, with $\Phi^{6-i}(y, z) = e^i z - y_i e^3$ $(i = 1, 2)$.

We will find six further solutions $T^1, \ldots, T^6$ which will later be given the meaning of point forces and torques. We combine the column vectors (3.1) in a 4×6 matrix $\mathbf{W}$ and define another matrix

$$\mathbf{V} = \left\| \begin{matrix} -z & 0 & 0 & 0 & 1 & 0 \\ 0 & -z & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{matrix} \right\| = (\mathbf{V}^1, \ldots, \mathbf{V}^6) \tag{3.2}$$

We note that $\mathbf{W}$ is obtained from $\mathbf{V}$ by permuting the outer columns and changing signs. Let $\mathbf{T}$ denote a matrix satisfying the equality

$$\mathbf{M}(z)\mathbf{D}(\partial_z)\mathbf{T}(z) = \mathbf{V}(z) \tag{3.3}$$

Since $\mathbf{M}(z) = Z(z)\mathbf{M}(1)Z(z)$, where $Z(z) = \operatorname{diag}\{z^{2\gamma}, z^{2\gamma}, z^{\gamma}, z^{2\gamma}\}$, we arrive at the formulae

$$\begin{aligned} &T^j(z) = \mathbf{D}(\partial_z)^{-1} Z(z)^{-1} M(1)^{-1} Z(z)^{-1} \mathbf{V}^j(z) \\ &\mathbf{D}(\partial_z)^{-1} = \operatorname{diag}\{\partial_z^{-1}\partial_z^{-1}, \ \partial_z^{-1}\partial_z^{-1}, \ \partial_z^{-1}\partial_z^{-1}\} \\ &\partial_z^{-1} z^{\tau} = (\tau + 1)^{-1} z^{\tau+1} \quad \text{for} \quad \tau \neq 1 \\ &\partial_z^{-1} z^{-1} = \ln z, \ \ \partial_z^{-1} \ln z = z(\ln z - 1) \end{aligned} \tag{3.4}$$

Replacing the columns w in formulae (2.10) and (2.13) by T^j and taking (2.1) into account, we obtain fields of displacements Ψ^j that generate in (1.4) and (1.5) residual terms which are small near the point Q. Let us determine the power orders of the components of the vector-valued functions T^j. Letting an asterisk stand for factors which are of no significance in the present context, we have

$$\begin{aligned} &T^q(z) = (z^{\mu_q} *, \ z^{\mu_q} *, \ z^{\mu_q - 1 + \gamma} *, \ z^{\mu_q - 1} *) \\ &\mu_1 = \mu_2 = 3 - 4\gamma, \ \ \mu_3 = 2 - 3\gamma, \ \ \mu_4 = \mu_5 = \mu_6 = 2 - 4\gamma \end{aligned} \tag{3.5}$$

4. GREEN'S FORMULA AND POINT FORCES

The following one-dimensional Green's formula holds

$$\begin{aligned} &\int_a^b \mathbf{v}'(z)\mathbf{L}(z, \partial_z)\mathbf{w}(z)dz = 2\mathbf{E}(\mathbf{v}, \mathbf{w}) - (\mathbf{W}(\partial_z)' \mathbf{v}(z))' \mathbf{N}(z, \partial_z)\mathbf{w}(z)\Big|_a^b \\ &\mathbf{E}(\mathbf{v}, \mathbf{w}) = \frac{1}{2}\int_a^b (\mathbf{D}(\partial_z)' \mathbf{v}(z))' \mathbf{M}(z)\mathbf{D}(\partial_z)' \mathbf{w}(z)dz \\ &\mathbf{N}(z, \partial_z) = \mathbf{V}(\partial_z)' \mathbf{M}(z)\mathbf{D}(\partial_z)' \end{aligned} \tag{4.1}$$

Letting $\mathbf{1}_m$ and $\mathbf{0}_m$ denote the $m \times m$ identity and zero matrices, one can verify by direct calculations that

$$\mathbf{N}(z,\ \partial_z)\mathbf{W}(z) = \mathbf{0}_6,\ \ (\mathbf{W}(\partial_z)^t\mathbf{W}(z))^t\mathbf{N}(z,\ \partial_z)\mathbf{T}(z) = \mathbf{1}_6 \tag{4.2}$$

In the context of the one-dimensional model, relations (4.2) mean that the solutions T^1, T^2, T^3 and T^4, T^5, T^6 define transverse and longitudinal forces and twisting and bending torques. It turns out that in the non-symmetric case analogous treatment of three-dimensional fields requires some readjustment of the basis $\{T^1, \ldots, T^6\}$. To verify this ,we link the one-dimensional Green's formula (4.1) with the usual three-dimensional formula. Consider the integrals

$$I_{pq}(z) = \int_{\omega(z)} \Phi^p(y,\ z)\sigma^{(3)}(\Psi^q;\ y,\ z)dy =$$

$$= \int_{\omega(z)} \Phi^p(y,\ z)D_1AD(\nabla)^t\Psi^q(y,\ z)dy,\quad p,\ q = 1, \ldots, 6 \tag{4.3}$$

$$\Psi^q(y,\ z) = \Psi^{-2,q}(z) + \Psi^{-1,q}(y,\ z) + \Psi^{0,q}(y,\ z) + \Psi^{1,q}(y,\ z) \tag{4.4}$$

The three-dimensional fields (4.4) are defined, in accordance with representation (2.1), by the solutions T^q. Specifically: $\Psi^{-2,q}$, $\Psi^{-1,q}$ and $\Psi^{0,q}$ are given by formulae (2.10) and (2.13) in which w has been replaced by T^j, and $\Psi^{1,q}$ is determined from problem (2.15) with $U^k = \Psi^{q,k}$, $k = -1, 0$.

Thus, for the aforementioned interpretation of the vectors $\Psi^1, \ldots, \Psi^6$, we must have an equality $I = \mathbf{1}_6 + o(1)$, where $I = (I_{pq})$ is a 6×6 matrix whose elements are the integrals (4.3).

By formulae (2.3) and (2.10), (2.12), (2.13), the column vectors of strains constructed from Ψ^q are

$$\varepsilon(\Psi^q) \equiv D^t(\nabla)\Psi^q = A(D_y^tX + Y)\mathbf{D}T^q + (D_y^t\Psi^{3,q} + D_z^t\Psi^{2,q}) + D_z^t\Psi^{3,q} \tag{4.5}$$

By expansion (4.5), we can express the integral (4.3) as a sum $I_{pq}^0 + I_{pq}^1 + I_{pq}^2$ and calculate its terms.

First let $p = 3, 4$. By (4.5), we have

$$I_{pq}^0(z) = \int_{\omega(z)} \Phi^p(y)^t D_1A(D_y^tX(y,z) + Y(y))dy\mathbf{D}T^q(z)$$

$$I_{4q}^1 = O(z^{\gamma-1}),\ \ I_{4q}^2 = O(z^{2\gamma-2})q = 3, 4$$

The orders of the infinitesimals I_{4q}^1 and I_{4q}^2 can be computed on the basis of representations (3.5) and the inequalities

$$\left|\frac{\partial}{\partial y_i^m \partial z^n}\Psi^{(p,q)}(y,\ z)\right| \le cz^{\mu_q + p(\gamma-1) - m\gamma - n} \tag{4.6}$$

$$m, n = 0, 1, \ldots,\ \ i = 1, 2.$$

Let Y^p denote a column of the matrix Y. Since $D_1^t\Phi^p(y) = Y^p(y)(p = 3, 4)$, formulae (2.3) and (3.4) imply that

$$I_{pq}^0(z) = (\mathbf{e}^p)^t\mathbf{M}(z)\mathbf{D}(\partial_z)T^q(z) = (\mathbf{e}^p)^t V^q(z) = \delta_{p,q} \tag{4.7}$$

$$p = 3, 4;\ q = 1, \ldots, 6$$

We now turn to the case $p = 1, 2$. By (4.6), we have $I_{pq}^2(z) = O(z^{\gamma-1})(q = 1, 2)$ and $I_{p3}^2(z) = O(z^{2\gamma-2})$. We assert that $I_{pq}^0 = 0(q = 1, \ldots, 6)$. Since $\Phi^p = e^p$, we obtain by (2.14) and (2.18)

$$\int_{\omega(z)} (e^p)^t D_1A(D_y^tX + Y)dy = \int_{\omega(z)} (D_y^te^3y_p)^t A(D_y^tX + Y)dy =$$

$$= -\int_{\omega(z)} (y_pe^3)^t D_yA(D_y^tX + Y)dy + \int_{\partial\omega(z)} (y_pe^3)^t D_\nu A(D_y^tX + Y)ds_y \tag{4.8}$$

Using relations (2.15) and (2.16), (2.18), we find that

$$I_{pq}^1(z) = \int_{\omega(z)} (e^p)^t D_1AH_q^1dy = \int_{\omega(z)} (D_y^te^3y_p)^t AH_q^1dy =$$

$$= -\int_{\omega(z)} (y_pe^3)^t D_yAH_q^1dy + \int_{\partial\omega(z)} (y_pe^3)^t D_\nu AH_q^1ds_y =$$

$$= \int (y_pe^3)^t\partial_zD_1AH_q^0dy - \int (y_pe^3)^t\nu_0D_1AH_q^1ds_y = \partial_z \int (y_pe^3)^t D_1AH_q^0dy \tag{4.9}$$

$$H_q^1 = D_y^1\Psi^{(3,q)} + D_z^t\Psi^{(2,q)}H_q^0 = (D_y^t X + Y)\mathbf{D}(\partial_z)T^q$$

Since $y_p D_1^t e^3 = -(e^p)^t Y(y) = -Y^p$ we have

$$I_{pq}^1(z) = -(\mathbf{e}^p)^t\partial_z \mathbf{M}(z)\mathbf{D}(\partial_z)T^q(z), \quad p=1,\ 2,\ \ q=1,\dots,6 \tag{4.10}$$

By (3.3) and (3.2), the column-vectors $\mathbf{M}(z)\mathbf{D}(\partial_z)T^q(z)$ are constant for $q = 3, \dots, 6$, and therefore $I_{pq} = 0$. In the case $q = 1, 2$, we have

$$I_{pq}^1(z) = (\mathbf{e}^p)^t\partial_z \mathbf{M}(z)\mathbf{D}(\partial_z)T^q(z) = (\mathbf{e}^p)^t\partial_z z\mathbf{e}^q = (\mathbf{e}^p)^t\mathbf{e}^q = \delta_{p,q}$$

Now let $p = 5, 6$. Using relations (3.5) and (4.6), we observe that

$$I_{pq}^2(z) = O(z^{\gamma-1}), \quad q=4,5,6; \quad I_{pq}^2(z) = O(z^{\gamma}), \quad q=1,2$$

$$I_{p3}^2(z) = O(z^{2\gamma-1}) I_{pq}^0(z) + I_{pq}^1(z) = \int_{\omega(z)} \{(e^i z -$$

$$-e^3 y_i)^t D_1 A H_0^q(z) - (e^3 y_i)^t D_1 A H_q^1(z)\} dz, \quad i = p-4$$

Using equality (4.8) and repeating the transformations (4.9), we deduce, by virtue of relations (3.3) and (3.2) with $p = 5, 6$, that

$$I_{pq}^0(z) + I_{pq}^1(z) = -z(\mathbf{e}^p)^t\partial_z \mathbf{M}(z)\mathbf{D}(\partial_z)T^q(z) + (\mathbf{e}^p)^t \mathbf{M}(z)\mathbf{D}(\partial_z)T^q(z) \tag{4.11}$$

Thus, the matrix I of integrals (4.3) may be written as follows, apart from infinitesimals of order $O(z^{\gamma-1})$

$$I = \begin{Vmatrix} \mathbf{1}_3 + \mathbf{L} & \mathbf{K} \\ \mathbf{0}_3 & \mathbf{1}_3 \end{Vmatrix}, \quad \mathbf{K} = (\mathbf{K}_{ij}), \quad \mathbf{L} = (\mathbf{L}_{ij})$$

$$\mathbf{L}_{3m} = I_{3m}^0, \quad \mathbf{L}_{33} = \mathbf{L}_{mj} = 0, \quad \mathbf{K}_{ij} = I_{i,j+3}, \quad m=1,2; \quad i,j=1,2,3$$

If the structure of the peak is symmetric, $\mathbf{K} = \mathbf{L} = \mathbf{0}_3$, so that the three-dimensional fields $\Psi^1, \dots, \Psi^6$ themselves admit of the same mechanical interpretation as $T^1, \dots, T^6$ (see the text after (4.2)). If $\mathbf{K}$ and $\mathbf{L}$ are not zero, we must invert the matrix I and introduce a new basis

$$(\Psi_0^1, \Psi_0^2, \dots, \Psi_0^6) = (\Psi^1, \dots, \Psi^6) I^{-1} =$$

$$= (\Psi^1, \dots, \Psi^6) \begin{Vmatrix} \mathbf{1}_3 - \mathbf{L} & \mathbf{K}(\mathbf{L} - \mathbf{1}_3) \\ \mathbf{0}_3 & \mathbf{0}_3 \end{Vmatrix} \tag{4.12}$$

Since the lower blocks of the matrix I^{-1} consist of zeros, the vectors Ψ_0^{j+3} and Ψ^{j+3} are not different ($j = 1, 2, 3$). The differences $\Psi_0^m - \Psi_m$ are linear combinations of the matrices $\Psi^3, \dots, \Psi^6$. Computation of the orders (with respect to z) of the coefficients of the combinations shows that the resulting increments are infinitesimal relative to Ψ^m. The same is true of the difference $\Psi_0^3 - \Psi^3$ expressed in terms of Ψ^1, Ψ^2 and Ψ^4, Ψ^5, Ψ^6. Thus, orthogonalization of the basis (4.4) implies changes only in the lower-order asymptotic terms.

5. SPECIFIC FORMULAE

Suppose an isotropic body (1.1) (with Lamé constants $\lambda \geqslant 0$ and $\mu > 0$) is formed by rotation of an arc of a circle of radius d about the tangent, that is, after dropping infinitesimals of order $o(z^2)$

$$Y(z) = 0, \quad \gamma = 2, \quad \omega = \{y \in \mathbf{R}^2 : |y| < d^{-1}\}$$

Suppose that $d = 1$. Then the matrices A, X and $\mathbf{M}$ are given by

$$A = \begin{Vmatrix} 2\mu\mathbf{1}_3 + \lambda\mathbf{U}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & 2\mu\mathbf{1}_3 \end{Vmatrix}$$

$$X(y) = \tau\begin{Vmatrix} \alpha^2(y_1^2 - y_2^2) & y_1 y_2 & -y_1 & 0 \\ y_1 y_2 & \alpha^2(y_2^2 - y_1^2) & -y_2 & 0 \\ 0 & 0 & 0 & 0 \end{Vmatrix}$$

$$\mathbf{M} = \mathrm{diag}\left\{\frac{\pi}{4}E, \frac{\pi}{4}E, 2\pi E, \frac{\pi}{4}\mu\right\}, \quad \nu = \frac{\lambda}{2(\lambda+\mu)}, \quad E = \mu\frac{3\lambda+2\mu}{\lambda+\mu}$$

where ν and E are Poisson's ratio and Young's modulus and $\mathbf{U}_3$ is the 3×3 matrix all of whose elements are 1's. The polynomial solutions (3.4) of the resulting problem (2.22) have the form

$$T^i(z) = -\pi^{-1}\beta_2^{-1}\beta_3^{-1}E^{-1}z^{\beta_3}\mathbf{e}^i$$

$$T^{4+i}(z) = \pi^{-1}\beta_1^{-1}\beta_2^{-1}E^{-1}z^{\beta_2}\mathbf{e}^i, \quad i = 1, 2$$

$$T^3(z) = \frac{1}{8}\pi^{-1}\beta_2^{-1}E^{-1}z^{\beta_2/2}\mathbf{e}^3, \quad T^4(z) = 2\pi^{-1}\beta_1^{-1}\mu^{-1}z^{\beta_1}\mathbf{e}^4$$

$$\beta_k = k - 4\gamma, \ k = 0, 1, 2, 3$$

By symmetry, the three-dimensional fields $\Psi^1, \ldots, \Psi^6$ are determined by formula (4.4), that is, the matrix I^{-1} in (4.12) is the identity. Thus, the singular solutions generated by unit shearing forces are defined as follows

$$\Psi^5(y,z) = \pi^{-1}E^{-1}(e^1(\beta_1^{-1}\beta_2^{-1}z^{\beta_2} + \alpha^2(y_1^2 - y_2^2)z^{\beta_0}) + e^2 y_1 y_2 z^{\beta_0})$$

$$\Psi^6(y,z) = \pi^{-1}E^{-1}(e^1 y_1 y_2 z^{\beta_0} + e^2(\beta_1^{-1}\beta_2^{-1}z^{\beta_2} + \alpha^2(y_2^2 - y_1^2)z^{\beta_0})) \tag{5.1}$$

The singularities $O(z^{1-3\gamma})$ corresponding to twisting and bending torques are weaker than the singularity $O(z^{2-4\gamma})$ of the vectors Ψ^5 and Ψ^6, namely

$$\Psi^4(y,z) = -2\pi^{-1}\mu^{-1}\beta_1^{-1}\alpha(e^1 y_2 z^{\beta_1} - e^2 y_1 z^{\beta_1})$$

$$\Psi^1(y,z) = -2\pi^{-1}E^{-1}(e^1(\beta_2^{-1}\beta_3^{-1}z^{\beta_3} + \alpha^2(y_1^2 - y_2^2)z^{\beta_1}) + e^2 y_1 y_2 z^{\beta_1})$$

$$\Psi^2(y,z) = -2\pi^{-1}E^{-1}(e^1 y_1 y_2 z^{\beta_1} + e^2(\beta_2^{-1}\beta_3^{-1}z^{\beta_3} + \alpha^2(y_1^2 - y_2^2)z^{\beta_1})) \tag{5.2}$$

Finally, the minimum singularity exponent $1 - 2\gamma$ is associated with a longitudinally acting force

$$\Psi^3(y,z) = -\frac{1}{8}\pi^{-1}E^{-1}(e^1 y_1 z^{-1+\beta_2/2} + e^2 y_1 z^{-1+\beta_2/2-1} - e^3 2\beta_2^{-1}z^{\beta_0/2}) \tag{5.3}$$

The singularities were compared on the basis of the orders of the displacements as $z \to 0$. The stresses and strains are $O(z^{-2\gamma-1})$ for all torques and $O(z^{-2\gamma})$ for all forces, that is, in this sense the longitudinal and transverse forces balance out. However, the exponents $-2\gamma-1$ and -2γ may be found by simpler reasoning and do not require a complete investigation of the structure of the elastic solutions.

Thus, the singularity of the displacement field near the tip of a symmetric isotropic peak proves to be a linear combination of the vector-valued functions (5.1)–(5.3). Similarly simple relations may be obtained for an arbitrary peak, non-symmetric or anisotropic, provided the matrix **M** in the system of differential equations (2.22) is known. Computation of the matrix $\mathbf{M}(z) = Z(z)\mathbf{M}(1)$ from the integral representation (2.23) is the only point in the asymptotic procedure where it is necessary to solve a two-dimensional problem of elasticity theory (2.14), over the cross-section $\omega = \omega(1)$. After that, the explicit formulae (3.4) yield the vectors $T^1, \ldots, T^6$, which depend on the variable z. Replacing the column-vectors ω by T^j in expressions (2.10)–(2.13) and using representation (2.1) for the three-dimensional field, one obtains singular solutions $\Psi^1, \ldots, \Psi^6$ analogous to (5.1)–(5.3). Finally, renormalizing these solutions in accordance with (4.12) gives them the same physical meaning as in the isotropic case.

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